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# On the Ideal and Radical Embedding of Algebras, I. Extreme Embeddings

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## INTRODUCTION

This paper and its sequel [5] offer a general approach to ideal embedding problems in algebra. This approach has recently grown out of the author's work [1–4] on the much more special (and still recalcitrant) *radical embedding problem*: given a nilpotent algebra $N$, finite-dimensional over the field $k$, describe the family of unital $k$-algebras $A$ with radical rad $A$ isomorphic to $N$. (For perhaps the earliest treatment of this problem, see Hall's 1940 paper [6].)

The particular question which led to the present generalization was: under what conditions and in what senses does there exist a unique unital $k$-algebra $E$ such that all solutions $A$ to the "equation" rad $A \simeq N$ are "essentially" subalgebras of $E$? (A typical example of this phenomenon is afforded by letting $N$ consist of all strictly upper triangular $n$ by $n$ matrices over $k$, and letting $E$ be the unital algebra of all upper triangular $n$ by $n$ matrices.)

This has proved to be a most fruitful question. As we progressed, it became apparent that many of the issues and methods generated should be discussed in a much less special context. Thus in the present work $k$ has become a commutative unital ring of scalars, while the finite-dimensional nilpotent $N$ has become an arbitrary $k$-algebra $I$, without finiteness assumptions, which we seek to embed as an ideal, or perhaps as a semidirect summand, or perhaps even as the Jacobson radical (if $I$ happens to be a radical algebra) in unital $k$-algebras $A$. For a fixed $I$, the class of such embeddings can be given the structure of a category and, more importantly for our purposes, the structure of a partially ordered class (modulo a natural equivalence relation reminiscent of homotopy equivalence). The quest for the algebras $E$ described in the second paragraph above becomes (Questions 1.6.2 below) the search for maximal elements, possibly unique, in this partially ordered class of embeddings. (Note that there is always a unique *minimal* embedding, afforded by "adjoining a unity!") By the

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way, we speak of "extreme" ideal embeddings in the above-mentioned ordering, rather than "maximal" embeddings, to avoid conflict with "maximal ideals," which is quite a different issue.

Here are some highlights of the present paper:

(1) a hierarchy of four natural equivalence relations (equality, isomorphism, stable equivalence, equivalence) in the class of ideal embeddings of $I$ (Section 1.3);

(2) the semidirect ideal embeddings of $I$ occupy the "bottom" of the ordered class of all ideal embeddings (Theorem 1.5.2);

(3) the surprising dichotomy that $I$ has either no or one extreme ideal embedding (up to equivalence) according as ann $I \neq (0)$ or ann $I = (0)$ (Theorems 1.7.1 and 2.3.1); in 1.7.1 we present a construction of "idealizing overalgebras" which may be of independent interest;

(4) the class of *semidirect* ideal embeddings of $I$ always has extreme elements, in contrast with (3) above (Theorem 2.5.2);

(5) if $I^2 = I$ or if ann $I = (0)$, then $I$ has a *unique* extreme semidirect ideal embedding, but these conditions are not necessary (Theorem 2.5.3);

(6) a theory of the "multiplier algebra" $k$-pairs $I$ of the algebra $I$.

This is a crucial invariant of $I$ in (3)–(5) above. (It is not the same as the "algebra of multiplications" [7, p. 46]). Only after developing this did the author learn that the multiplier algebra is well known to analysts (the name is theirs) who work with $C^*$-algebras; see the recent memoir [8] of Lazar and Taylor.

The present paper is divided into two halves. In the first half we develop the classes of ideal embeddings of $I$ and describe them as fully as we can without the multiplier algebra $k$-pairs $I$. In the second half, we introduce $k$-pairs $I$ and get the second result of (3) and (4)–(6). There are many examples throughout, some exercises, and several open questions.

In the sequel to this paper the author will consider radical algebras $I$, in particular nilpotent $I$, and their radical embeddings, using the multiplier algebra $k$-pairs $I$. The results to be given there will actually include those found earliest in the course of the present investigation.

We mention that this paper is almost entirely self-contained and does not require knowledge of [1–4].

## 1. Basic Theory

### 1.1 *Classes of Ideal Embeddings*

We fix the following notation:

$k$ = a commutative unital ring of scalars;

$I$ = an associative $k$-algebra, not necessarily unital;

$k$-Alg = the category of associative $k$-algebras, not necessarily unital, and $k$-algebra homomorphisms;

$k$-$\mathrm{Alg}_1$ = the "unital" subcategory consisting of unital associative $k$-algebras and homomorphisms preserving the unity elements;

$\mathrm{ann}(X; B)$ = the two-sided annihilator of subset $X$ of a $B$-bimodule.

We will be interested throughout in certain mappings of $I$ into unital $k$-algebras.

1.1.1 DEFINITION. A pair $(A, \alpha)$ is an *ideal submersion* of $I$ if (i) $A$ is a unital $k$-algebra, that is, an object in $k$-$\mathrm{Alg}_1$, (ii) $\alpha: I \to A$ is a homomorphism in $k$-Alg whose image $\alpha I$ is an ideal (i.e., two-sided) in $A$. Most important for us, the ideal submersion $(A, \alpha)$ is an *ideal embedding* if it satisfies (iii) $\alpha$ is injective.

Note that even if $I$ happens to have a unity $1_I$, we do not require $\alpha(1_I) = 1_A$ in (ii). Of course, $\alpha(1_I)$ will be an idempotent in $A$.

An immediate example of ideal embedding: the pair $(k \dotplus I, \iota)$ produced by "adjoining a unity" to $I$ (even if $I$ already has a unity). Here $k \dotplus I$ is the familiar external semidirect sum with multiplication $(c, x)(c', x') = (cc', cx' + c'x + xx')$, and $\iota(x) = (0, x)$.

*Notation.* The symbol $\dotplus$ will denote $k$-direct sum.

Another instance of ideal embedding: the pair $(I, \mathrm{id})$, with id = identity map, is an ideal embedding of $I$ iff $I$ is unital. See 1.4.3 for unital $I$.

We will be especially interested in "semidirect" and "radical" ideal embeddings. Let us define these now.

1.1.2 DEFINITION. An ideal embedding $(A, \alpha)$ of $I$ is *semidirect* if the ideal $\alpha I$ of $A$ is a semidirect summand of $A$ in the sense that $A = S \dotplus \alpha I$, where $S$ is a subalgebra of $A$ in $k$-Alg.

One checks here that $S$ will be unital because $A$ is. However, we do not require that $1_S = 1_A$. For example, if $I$ is unital then any external direct sum $A = S \oplus I$ with unital $S$ clearly yields a semidirect embedding of $I$ such that $1_A = 1_S + 1_I$. This is true even if $S = (0)$, which yields $(I, \mathrm{id})$.

Note that, thanks to $(k \dotplus I, \iota)$, every algebra $I$ has at least one semidirect embedding. It is now easy to produce many more: all $((k \dotplus I) \oplus C, \iota \oplus 0)$ with $C$ unital but otherwise arbitrary and $0: I \to C$ the zero map.

1.1.3 DEFINITION. An ideal embedding $(A, \alpha)$ of $I$ is a *radical embedding* if $\alpha I = \mathrm{rad}\, A$ = the Jacobson radical of $A$.

It is clear that $I$ admits radical embeddings only if $I$ is itself a radical $k$-algebra, that is, $I = \mathrm{rad}\, I$.

These definitions prompt some further notation of frequent use:

$\mathscr{IE}(I)$ = the class of ideal embeddings of $I$;

$\mathscr{SD}(I)$ = the subclass of $\mathscr{IE}(I)$ consisting of all semidirect ideal embeddings of $I$;

$\mathscr{RE}(I)$ = the subclass of $\mathscr{IE}(I)$ consisting of all radical embeddings of $I$.

1.1.4 GENERAL PROBLEM. Obtain satisfying descriptions of $\mathscr{IE}(I)$, of $\mathscr{SD}(I)$ and, if $I$ is radical, of $\mathscr{RE}(I)$ for interesting or appropriate $k$-algebras $I$. The original radical embedding problem [1–3, 6] was the description of $\mathscr{RE}(N)$ in the case $N$ was a finite-dimensional nilpotent algebra over a field $k$.

## 1.2 *Morphisms of Ideal Submersions*

We will use morphisms to compare ideal embeddings (and occasionally submersions) of $I$. In particular, morphisms will be used in 1.3 to define certain important equivalence relations in $\mathscr{IE}(I)$ and its subclasses.

1.2.1 DEFINITION. Let $(A, \alpha)$ and $(B, \beta)$ be ideal submersions of $I$. A *morphism* from $(A, \alpha)$ to $(B, \beta)$ is a pair $(f, \phi)$ where (i) $f$ is an automorphism of the $k$-algebra $I$, (ii) $\phi$: $A \to B$ is a homomorphism in $k$-Alg, and (iii) the diagram

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[d, "\phi"] \\ I \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes in $k$-Alg. A morphism $(f, \phi)$ is an *isomorphism* if (iv) $\phi$ is an isomorphism in $k$-Alg (whence in $k$-$\mathrm{Alg}_1$).

*Notation.* We write $(f, \phi) : (A, \alpha) \to (B, \beta)$ for morphisms. If such a morphism exists, then we frequently write $(A, \alpha) \prec (B, \beta)$. Moreover, if $(A, \alpha)$ and $(B, \beta)$ are isomorphic in the sense of 1.2.1 (iv), then we write $(A, \alpha) \cong (B, \beta)$.

*Preview.* Two other relations, $\approx$ and $\sim$, will be defined in 1.3 below.

*Comment.* It is important to the theory that, in the above definition of morphism, the map $\phi$ is not required to send $1_A$ to $1_B$, despite the fact that both $A$ and $B$ are unital, nor is $\phi$ required to be injective or surjective. However, if $(A, \alpha)$ and $(B, \beta)$ are ideal *embeddings* of $I$, then the map $\phi$ must be nondegenerate in the sense that it maps the ideal $\alpha I$ of $A$ isomorphically onto the ideal $\beta I$ of $B$.

*Caution.* $(A, \alpha) \prec (B, \beta)$ does not imply, in case $k$ is a field, that $k$-dim $A \leqslant k$-dim $B$.

1.2.2 EXAMPLE. Suppose $(A, \alpha)$ is in $\mathcal{IE}(I)$ and that $M$ is an ideal of $A$ "disjoint" from $\alpha I$ in the sense that $\alpha I \cap M = (0)$. (Thus, $M \subset \mathrm{ann}(\alpha I; A) =$ the two-sided annihilator of $\alpha I$ in $A$.) Let $B = A/M$, let $\pi: A \to B$ be the natural map, and let $\beta = \pi\alpha$. Then there is a morphism $(\mathrm{id}, \pi)$: $(A, \alpha) \to (B, \beta)$ in $\mathcal{IE}(I)$.

Note that if $I \neq (0)$ and if $M$ is chosen (using Zorn's lemma, if need be) as being maximal with respect to the condition $\alpha I \cap M = (0)$, then $\beta I$ is an "essential" ideal of $B$ in the sense that $\beta I$ has nonzero intersection with every nonzero ideal of $B$. Thus we have that, in $\mathcal{IE}(I)$, every $(A, \alpha) \prec (B, \beta)$ where $\beta I$ is essential in $B$.

Note in particular that if $I \neq (0)$ and $\mathrm{ann}\, I = \mathrm{ann}(I; I) =$ the two-sided annihilator of $I$ in $I$ is zero, then $M = \mathrm{ann}(\alpha I; A)$ is the *unique* ideal of $A$ such that $(A, \alpha) \prec (B, \beta)$ and $\beta I$ is essential in $B$.

1.2.3 COMMENT ON CATEGORIES. The notion of morphism allows us to endow $\mathcal{IE}(I)$ with the structure of category. We slight the category approach, however, in favor of organizing $\mathcal{IE}(I)$ and its subclasses in terms of the relation $\prec$ and various equivalence relations to be defined next.

### 1.3 *Equivalence Relations on Ideal Embeddings*

Let $\mathscr{C}$ be a subclass of $\mathcal{IE}(I)$. Thus far we have two equivalence relations in $\mathscr{C}$, namely equality in the usual sense and isomorphism. These are very strong, and in our efforts to organize $\mathscr{C}$ we will make considerable use of two weaker equivalence relations, to be called "stable equivalence" and "equivalence". Let us turn to these now.

1.3.1 DEFINITION. Two embeddings $(A, \alpha)$ and $(B, \beta)$ in a subclass $\mathscr{C}$ of $\mathcal{IE}(I)$ are ***stably equivalent*** in $\mathscr{C}$ if there exist unital $k$-algebras $C$, $D$ such that the embeddings $(A \oplus C, \alpha \oplus 0)$ and $(B \oplus D, \beta \oplus 0)$ are in $\mathscr{C}$ and are isomorphic embeddings of $I$ in the sense of 1.2.1 (iv).

*Notation.* We write $(A, \alpha) \approx (B, \beta)$ to denote stable equivalence.

In this definition $A \oplus C$ is the usual external direct sum, $0: I \to C$ is the zero map, etc. Basically we are forming the "direct sum" of $(A, \alpha)$ and the "zero submersion" $(C, 0)$. We remark in passing that the obvious definition of direct sum of two ideal *embeddings* need not yield an ideal embedding.

Note that "stable equivalence" depends on the subclass $\mathscr{C}$. Thus, we might wish to require $A \oplus C$ to be finite dimensional, or require $C$ to be semisimple (for radical embeddings) and so on.

Finally, we define what will be our weakest equivalence relation (and our most important).

1.3.2 Definition. $(A, \alpha)$ and $(B, \beta)$ in $\mathscr{C}$ are *equivalent* if both $(A, \alpha) \prec (B, \beta)$ and $(B, \beta) \prec (A, \alpha)$. See 1.2.1.

*Notation.* We write $(A, \alpha) \sim (B, \beta)$ if the two ideal embeddings are equivalent.

*Caution.* The relation $\prec$ does *not* define a partial ordering on $\mathscr{IE}(I)$. The relation is reflexive and transitive, but it is not antisymmetric on $\mathscr{IE}(I)$. For it is easy to concoct ideal embeddings such that $(A, \alpha) \prec (B, \beta)$ and also $(A, \alpha) \succ (B, \beta)$ yet $(A, \alpha) \neq (B, \beta)$, as is required for partial ordering. In fact Example 1.4.7 shows equivalence need not imply stable equivalence. If $\mathscr{C}$ is any subclass of $\mathscr{IE}(I)$, however, then the quotient class $\mathscr{C}/\sim$ *is* partially ordered by the obvious notion of $\prec$ in $\mathscr{C}/\sim$. Nonetheless, we usually prefer to work in $\mathscr{C}$ rather than in its partially ordered quotient $\mathscr{C}/\sim$.

Let us compare our four equivalence relations on ideal embeddings of $I$. It is a fact that

$$= \text{ implies } \cong \text{ implies } \approx \text{ implies } \sim$$

in any class $\mathscr{C}$ of ideal embeddings. Only the last implication is not immediate. We establish this below (Corollary 1.3.4) after answering the question, "What do stably equivalent embeddings of $I$ have in common?"

1.3.3 Theorem. *Let $(A, \alpha)$ and $(B, \beta)$ be stably equivalent in $\mathscr{IE}(I)$. Then there exist ideal direct sum decompositions $A = A_1 \oplus A_0$ and $B = B_1 \oplus B_0$ such that*

(i) $\alpha I \subset A_1$ *and* $A_0 \subset \operatorname{ann}(\alpha I; A)$;

(ii) *likewise,* $\beta I \subset B_1$ *and* $B_0 \subset \operatorname{ann}(\beta I; B)$;

(iii) *the ideal embeddings $(A_1, \alpha_1)$ and $(B_1, \beta_1)$ are isomorphic where, for $x$ in $I$, we define $\alpha_1(x) = \alpha(x)$ in $A_1$ and $\beta_1(x) = \beta(x)$ in $B_1$.*

*Proof.* Let $(f, \phi)\colon (A \oplus C, \alpha \oplus 0) \to (B \oplus D, \beta \oplus 0)$ be the isomorphism. To decompose $A$, we use the decomposition of $A \oplus C$ obtained from $\phi^{-1}$, that is,

$$A \oplus C = \phi^{-1}(B \oplus D) = \phi^{-1}(B \oplus (0)) \oplus \phi^{-1}((0) \oplus D).$$

Since the two summands on the right-hand side here are unital ideals, it is straightforward that the ideal $A \oplus (0)$ in $A \oplus C$ also decomposes,

$$A \oplus (0) = (A \oplus (0)) \cap \phi^{-1}(B \oplus (0)) \oplus (A \oplus (0)) \cap \phi^{-1}((0) \oplus D).$$

This is transferred into an internal direct sum decomposition $A = A_1 \oplus A_0$

by the obvious isomorphism $A \to A \oplus (0)$. Assertion (i) follows from the definitions of "stable equivalence" and "isomorphism of ideal embeddings."

By symmetry we get $B = B_1 \oplus B_0$ and assertion (ii).

Finally note that the restriction $\phi_1$ of $\phi$ to $A_1$ is an isomorphism onto $B_1$ and so yields the isomorphism $(f, \phi_1)$: $(A_1, \alpha_1) \to (B_1, \beta_1)$ of assertion (iii). This completes the proof of the theorem.

Now we reassure ourselves that stable equivalence is indeed a special case of our equivalence.

1.3.4 Corollary. *If* $(A, \alpha) \approx (B, \beta)$, *then* $(A, \alpha) \sim (B, \beta)$.

*Proof.* As in the preceding proof, let $(f, \phi)$ be the given isomorphism of $(A \oplus C, \alpha \oplus 0)$ onto $(B \oplus D, \beta \oplus 0)$. To show $(A, \alpha) \prec (B, \beta)$, define a morphism $(g, \psi)$: $(A, \alpha) \to (B, \beta)$ as follows. Take $g = f$, and define $\psi$ by

$$\psi = \phi_1 \oplus 0: A = A_1 \oplus A_0 \to B_1 \oplus B_0 = B,$$

where $\phi_1$ is the restriction of $\phi$ to $A_1$ as in the preceding proof. Note that ker $\psi = A_0$ and im $\psi = B_1$.

The argument that $(B, \beta) \prec (A, \alpha)$ is entirely symmetric. This completes the proof of the corollary.

## 1.4 *Examples: Embeddings, Morphisms, and Equivalences*

These will show, among other things, that the four equivalence relations $=$, $\simeq$, $\approx$, and $\sim$ are distinct and necessary for a comprehensive theory.

The moral of Example 1.4.3 below is that the thories of $\mathscr{IE}(I)$ and $\mathscr{SD}(I)$ relative to the relation $\approx$ of stable equivalence are trivial if the algebra $I$ happens to be unital (in particular, if $I = (0)$).

1.4.1 Example. If $(A, \alpha)$ is in $\mathscr{IE}(I)$, $f$ is an automorphism of $I$, and $\phi$: $A \to B$ an isomorphism in $k$-$\mathrm{Alg}_1$, then $(B, \phi\alpha f^{-1})$ is an embedding of $I$ isomorphic to $(A, \alpha)$.

1.4.2 Example. To construct stably equivalent $(A, \alpha)$ and $(B, \beta)$ which are not isomorphic, let $(E, \epsilon)$ be an ideal embedding of $I$ and $C$, $D$ unital $k$-algebras. Define $(A, \alpha) = (E \oplus D, \epsilon \oplus 0)$ and $(B, \beta) = (E \oplus C, \epsilon \oplus 0)$ and observe $(A \oplus C, \alpha \oplus 0) \simeq (B \oplus D, \beta \oplus 0)$, giving a stable equivalence $(A, \alpha) \approx (B, \beta)$. Clearly $C$ and $D$ may be chosen (if the class $\mathscr{C}$ is sufficiently large) so that $A$ is not isomorphic to $B$.

1.4.2 Example: $I$ Unital. Let $I$ have unity element $1_I$, so that $(I, \mathrm{id})$ is in $\mathscr{IE}(I)$, where id $=$ the identity map. Let $(A, \alpha)$ be any ideal embedding of $I$.

One checks that the element $e = 1_A - \alpha(1_I)$ is an idempotent *central* in $A$, whence we obtain by standard arguments an ideal direct sum decomposition $A = eAe \oplus \alpha I$. It follows that $(A, \alpha) \cong (I \oplus eAe, \text{id} \oplus 0)$, so that $(A, \alpha) \approx (I, \text{id})$. We state this as a theorem.

THEOREM. *If the k-algebra I is unital, then all ideal embeddings of I are stably equivalent.*

1.4.4 EXERCISES. The converse to the preceding theorem is also true. *If all ideal embeddings of I are stably equivalent, then I is unital.* Hint: suppose $I$ is not unital, and construct $(B, \beta)$ not stably equivalent to $(k \dotplus I, \iota)$. See 1.4.7.

However, this last implication is false if we replace "stably equivalent" by "equivalent." Hint: let $I$ be an algebra of all polynomials with constant term zero. See 1.6.6 and 2.1.4.

For which $I$ does $\mathscr{I}\mathscr{E}(I)/\sim$ reduce to a point?

1.4.5 COMMENT. As $I$ becomes more "degenerate" in various senses (such as nonunital, or $I^2 < I$, or ann $I \neq (0)$, or $I$ nilpotent, for example), the description of $\mathscr{I}\mathscr{E}(I)$ becomes more complicated than the simple statement of 1.4.3.

1.4.6 COMMENT. Consider the problem of describing $\mathscr{R}\mathscr{E}(I)$, the class of all radical embeddings of a given radical algebra $I$. In fact, consider the case $I = (0)$. By 1.4.3, all radical embeddings of $I = (0)$ are stably equivalent. This "solution" to the radical embedding problem for $I = (0)$ is of course not yet satisfactory. For one now wants in this context a description of all Jacobson-semisimple $k$-algebras *up to isomorphism*. This of course has been one of the major programs of algebra for the past century. It is generally hopeless unless the scalars $k$ form a "very nice" field. But that is not our problem here.

1.4.7 EXAMPLE. We show how to construct ideal embeddings which are equivalent but not stably equivalent. This serves to justify our introduction of the relation $\sim$; the reader will agree that the two embeddings we construct are basically the same from the point of view of $I$, and therefore should be identified by the theory in some way.

Select a $k$-algebra $I$ and an ideal embedding $(A, \alpha)$ such that $A$ is directly indecomposable (i.e., into two-sided ideals) and also $\alpha I \neq A$. (We include a concrete example below.)

Next we obtain $(B, \beta)$. We select a nonzero $A$-bimodule $M$ which is "unital" in the sense that $1_A \cdot m = m = m \cdot 1_A$ for all $m$ in $M$ and also is such that $\alpha I$ is contained in the two-sided annihilator of $M$. Construct the familiar external semidirect sum $B = A \dotplus M$ with multiplication $(a, m) \times (a', m') = (aa', am' + ma')$. Note that $M^2 = (0)$ in $B$. Note also that $B$ is directly indecomposable (because $A$ is). Now $(B, \beta)$ is an ideal embedding of $I$, where $\beta(x) = (\alpha(x), 0)$ for $x$ in $I$.

It is straightforward that $(A, \alpha) \sim (B, \beta)$. Now assume stable equivalence $(A, \alpha) \approx (B, \beta)$. By Theorem 1.3.3 and the direct indecomposability of both $A$ and $B$, one concludes $A$ and $B$ are isomorphic as $k$-algebras. If the module $M$ has been wisely chosen beforehand so that $A$ and $B = A \dotplus M$ are not isomorphic then the assumption of stable equivalence is false.

For a concrete example, take $k =$ an integral domain, $A = k[t] =$ the usual polynomial algebra, $I = tk[t]$, and $M = k \dotplus \cdots \dotplus k$, with $M^2 = (0)$ and $tM = Mt = (0)$.

See below for an example with $M^2 \neq (0)$.

1.4.8 Example. Let $k = \mathbb{Z}$, $I = p^n\mathbb{Z}$ for some prime $p$, $n \geqslant 1$, $A = \mathbb{Z}$, $\alpha: I \to A$ the inclusion map.

To construct $(B, \beta)$, form the $\mathbb{Z}$-algebra $\mathbb{Z}/p^{n+1}\mathbb{Z}$ and let $M$ be the nilpotent ideal generated by $p + p^{n+1}\mathbb{Z}$. Let $B = A \dotplus M$ with the natural action of $A = \mathbb{Z}$ on $M$ and $M^{n+1} = (0)$; this is the usual adjunction of a unity to $M$. Let $\beta: I \to B$ be the inclusion, $\beta(x) = (\alpha(x), 0)$ for $x$ in $I$.

Just as in 1.4.7, one checks that $(A, \alpha) \sim (B, \beta)$ in $\mathscr{IE}(I)$, but that $(A, \alpha)$ and $(B, \beta)$ are not stably equivalent. For this would force $A$ and $B$ to be isomorphic, whereas $B$ has a nontrivial nilpotent ideal $(0) \dotplus M$.

1.4.9 Example. Let $k = \mathbb{Q}$ and $I = (0)$. Consider the ideal embeddings $(\mathbb{C}, 0)$ and $(\mathbb{C}(X), 0)$ where $X$ is a finite or countable nonempty set of indeterminates.

The standard inclusion $\mathbb{C} \to \mathbb{C}(X)$ yields $(\mathbb{C}, 0) \prec (\mathbb{C}(X), 0)$. Now recall the curious fact that there is a $\mathbb{Q}$-algebra monomorphism $\mathbb{C}(X) \to \mathbb{C}$ onto a proper (not algebraically closed!) subfield of $\mathbb{C}$. Thus $(\mathbb{C}, 0) \sim (\mathbb{C}(X), 0)$. This was predicted by 1.4.3 but here we have $I$ embedded as a maximal ideal and, moreover, both morphisms giving the equivalence are injective, yet neither is surjective.

1.4.10 Exercise. Let $(f, \phi): (A, \alpha) \to (B, \beta)$ be a morphism in $\mathscr{IE}(I)$. If $\phi$ is injective and $\phi(1_A)$ is a central idempotent in $B$, then $(A, \alpha) \approx (B, \beta)$.

## 1.5 *Equivalence and Semidirectness*

Here we point out briefly that the subclass $\mathscr{SD}(I)$ consisting of semidirect ideal embeddings is surprisingly well situated, with respect to $\sim$, in the class $\mathscr{IE}(I)$ of all ideal embeddings.

The following lemma is an elementary exercise in our definitions and linear algebra over $k$.

1.5.1 Lemma. *Let $(f, \phi): (A, \alpha) \to (B, \beta)$ be a morphism of ideal embeddings.*

*If $(B, \beta)$ is in $\mathscr{S}\mathscr{D}(I)$, say $B = T \dotplus \beta I$, then $(A, \alpha)$ is also in $\mathscr{S}\mathscr{D}(I)$; in fact, $A = \phi^{-1}(T) \dotplus \alpha I$. Thus, any ideal embedding which is $\prec$ a semidirect embedding is itself semidirect.*

Here is our result, an immediate consequence of the lemma.

1.5.2 THEOREM. *The property of semidirectness in $\mathscr{I}\mathscr{E}(I)$ is invariant under the equivalence relation $\sim$. Thus $\mathscr{S}\mathscr{D}(I)$ is a union of certain $\sim$-equivalence classes in $\mathscr{I}\mathscr{E}(I)$.*

It is not in general true that all semidirect embeddings of $I$ are equivalent. For example, let $I = k \dotplus \cdots \dotplus k$ be the free $k$-module of rank $n \geq 2$ with $I^2 = (0)$. If $n = rs$, then one can embed $I$ as the "upper right-hand corner" of size $r$ by $s$ in the $k$-subalgebra $A$ consisting of appropriate block upper triangular matrices (two diagonal blocks: the first $r$ by $r$, the second $s$ by $s$) inside the full matrix algebra $M(r + s; k)$. This affords a semidirect embedding of $I$. If $n$ is highly composite, then one gets many inequivalent embeddings from the many factorizations $n = rs$, $1 \leq r \leq s$.

We remark that, if the scalar ring $k$ is semisimple, then the above embeddings of $I$ are in fact radical embeddings. These we described in greater detail in [1, Theorem 4.1].

1.5.3 DISCUSSION. If one regards $\mathscr{I}\mathscr{E}(I)$ as ordered by the transitive relation $\prec$, then 1.5.1 says that the "lesser" embeddings of $I$ tend to be semidirect; a nonsemidirect embedding cannot be $\prec$ than a semidirect embedding. This is particularly striking in the familiar embedding $(k \dotplus I, \iota)$ obtained by adjoining a unity to $I$. This embedding is semidirect and, moreover, "injects" into every embedding of $I$ provided $I$ is nonunital. If $I$ is unital, then $(I, \mathrm{id})$ is a semidirect embedding, clearly $\prec$ than (and, in view of 1.4.3, stably equivalent to) all other embeddings of $I$.

### 1.6 *Extreme Embeddings*

As mentioned in the introduction, it is the concept of "extreme" ideal (especially radical) embedding which prompted the present investigations. We continue to let $I$ be a $k$-algebra and $\mathscr{C} \subset \mathscr{I}\mathscr{E}(I)$ a class of ideal embeddings of $I$.

1.6.1 DEFINITION. An embedding $(E, \epsilon)$ belonging to $\mathscr{C}$ is *extreme in* $\mathscr{C}$ if $(E, \epsilon) \prec (B, \beta)$ in $\mathscr{C}$ implies that $(B, \beta) \prec (E, \epsilon)$ as well, so that $(E, \epsilon) \sim (B, \beta)$.

The reader will mark the resemblence here to the notion of "maximal element" in a partially ordered set (which our $\mathscr{I}\mathscr{E}(I)$ is not!). However, the word "maximal" already has a meaning in ideal theory!

One readily checks that an embedding in $\mathscr{C}$ equivalent to an extreme embedding in $\mathscr{C}$ is itself extreme. It follows that the $\sim$-equivalence class of an extreme embedding in $\mathscr{C}$ does constitute a maximal element in the partially ordered quotient class $\mathscr{C}/\sim$. However, we prefer to work in $\mathscr{C}$ rather than in $\mathscr{C}/\sim$.

The following questions are central to Problem 1.1.4 of describing the classes $\mathscr{I}\mathscr{E}(I)$, $\mathscr{S}\mathscr{D}(I)$, etc. See 1.6.3 below.

1.6.2 QUESTIONS. Given $I$ and $\mathscr{C} \subset \mathscr{I}\mathscr{E}(I)$, does $\mathscr{C}$ have extreme members? And, in the class $\mathscr{C}$, is every embedding $\prec$ than an extreme embedding $(E, \epsilon)$?

We will find that the answers to 1.6.2 are sometimes negative. See 1.7.1.

1.6.3 COMMENT. The point of an affirmative answer to the second question in 1.6.2 is this. If every $(A, \alpha)$ belonging to $\mathscr{C}$ is $\prec$ than an extreme $(E, \epsilon)$ in $\mathscr{C}$, then knowledge of the extreme embeddings enables us to describe all $(A, \alpha)$ in $\mathscr{C}$. For $A$ will therefore be isomorphic to a subalgebra of an "extreme" $E$, or will at worst be an extension, by an ideal which annihilates $\alpha I$, of a subalgebra of $E$. Thus the extreme $(E, \epsilon)$ and certain of the subalgebras of $E$ hold the key.

1.6.4 COMMENT. In our program of describing $\mathscr{I}\mathscr{E}(I)$ with respect to the relation $\prec$, we saw in 1.5 that the "lesser" embeddings are semidirect. In studying the extreme embeddings now, we are looking at the "higher" end of $\mathscr{I}\mathscr{E}(I)$.

Here are some situations in which extreme embeddings do exist. See also Sections 2.3 and 2.5.

1.6.5 EXAMPLE: UNITAL IDEALS. If $I$ is unital, then every embedding is extreme in $\mathscr{I}\mathscr{E}(I)$ or in any containing subclass $\mathscr{C}$ by 1.4.3.

1.6.6 EXAMPLE: CERTAIN PRINCIPAL IDEALS. Suppose $I$ is a two-sided ideal in a unital $k$-algebra $A$ such that (i) $I = zA = Az$ for some $z$ in $I$, and (ii) $z$ is neither a left- nor a right-zero-divisor in $A$. Then we assert and will prove below that $(A, \alpha)$ is an extreme embedding in $\mathscr{I}\mathscr{E}(I)$, where $\alpha\colon I \to A$ is the inclusion.

*Comment.* This result applies to the familiar commutative principal ideal embeddings $n\mathbb{Z} \to \mathbb{Z}$, $p(t)\,k[t] \to k[t]$ with $k$ an integral domain, etc.

Moreover, to construct an example of $I$, $A$ in which $z$ is *not* central in $I$, let $A = C[z]$ where $C$ is a unital $k$-algebra with a nonidentity automorphism $\theta$, $z$ is an indeterminate over $C$, and where we define $cz = z\theta(c)$ for $c$ in $C$. Now let $I = Az = zA$ as usual.

In fact, the above construction is general in the sense that if $A$, $I$, $z$ satisfy (i) and (ii), then there is an automorphism $\theta\colon A \to A$ determined by the equation $az = z\theta(a)$. That $\theta(a_1a_2) = \theta(a_1)\,\theta(a_2)$ follows from the associativity of $A$. We use this below.

Now we prove the assertion that $(A, \alpha)$ is an extreme embedding of $I$. Suppose

$(f, \phi)$: $(A, \alpha) \to (B, \beta)$ is a morphism in $\mathscr{I}\mathscr{E}(I)$. Note that $\phi$ must here be injective.

We will define a morphism $(g, \psi)$: $(B, \beta) \to (A, \alpha)$ as follows. Define $g = f^{-1}$. To define $\psi$, check that for $b$ in $B$ there exists a unique $a$ in $A$ such that $b\phi(z) = \phi(az)$ in $\beta I$. Define $\psi(b) = a$. To verify that the function $\psi$ is multiplicative, note that $\phi(\psi(b_1b_2)z) = b_1b_2\phi(z) = b_1\phi(\psi(b_2)z) = b_1\phi(z\theta\psi(b_2)) = b_1\phi(z)\phi\theta\psi(b_2) = \phi(\psi(b_1)z)\phi\theta\psi(b_2) = \phi(\psi(b_1)z\theta\psi(b_2)) = \phi(\psi(b_1)\psi(b_2)z)$, whence $\psi(b_1b_2) = \psi(b_1)\psi(b_2)$ by the injectivity of $\phi$ and the fact that $z$ is not a right-zero-divisor in $A$.

To prove that $(f^{-1}, \psi)$ is a morphism, we must check that $\alpha f^{-1} = \psi\beta$ on $I$, that is, $\alpha = \psi\beta f = \psi\phi\alpha$. But actually $\psi\phi =$ identity on $A$; this follows from the fact that $\phi(az) = \phi(a)\phi(z) = \phi(\psi\phi(a)z)$. This completes the proof of the assertion.

*Preview.* In Example 2.1.5 we will see that the $A$ here is determined by $I$, $z$, and conditions (i), (ii) as part of a more general theory. See also Theorem 2.3.1.

## 1.7 *A Theorem on Algebras with Nonzero Annihilator*

The following somewhat surprising result is our first step in showing that the structure of $\mathscr{I}\mathscr{E}(I)$ depends strongly on the annihilator of $I$ in $I$. See 1.7.2 below.

1.7.1 THEOREM. *Let* ann $I \neq (0)$. *Then the class* $\mathscr{I}\mathscr{E}(I)$ *of all ideal embeddings of* $I$ *has no extreme elements.*

In brief, given an ideal embedding $(A, \alpha)$, we will inject $A$ into an algebra $Y$ which is larger "in a nontrivial sense" and in which the image of $I$ is still an ideal. It will follow that $(A, \alpha)$ is not extreme.

Thus, both questions in 1.6.2 have negative answers in case ann $I \neq (0)$ and $\mathscr{C} = \mathscr{I}\mathscr{E}(I)$.

1.7.2 PREVIEW. The result stated above should immediately be set beside two other facts, both known to the author well before Theorem 1.7.1.

First, Theorem 1.7.1 may be false for these same $I$ if we replace $\mathscr{I}\mathscr{E}(I)$ by some other subclass of interest, such as semidirect embeddings or radical embeddings if $I$ happens to be a radical algebra. We treat these cases in Section 2.5 and in the sequel to this paper.

Second, we will see in Theorem 2.3.1 that in the complementary case ann $I = (0)$, there does exist an embedding extreme in $\mathscr{I}\mathscr{E}(I)$; in fact there exists only one, which dominates (in the sense of $\prec$) all other ideal embeddings of $I$!

*Proof of the theorem.* We may suppose $I$ is an ideal in a unital $k$-algebra $A$. Let $x_0$ be an element of ann $I = \text{ann}(I; I)$. Let $J$ be a nonempty index set to be specified below. We construct a unital $k$-algebra $Y = Y(A, I, x_0, J)$ as follows. As a $k$-module, $Y$ consists of these 2 by 2 matrices:

$$Y = Y(A, I, x_0, J) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where $Y_{11} = A$, $Y_{22} = k$, and the "corners" $Y_{12}$ and $Y_{21}$ will now be defined. For each index $j$ in $J$, let $Y_{12j}$ be $A/I$, considered as a left $A$-, right $k$-module; that is, $Y_{12j} = {}_A(A/I)_k$. Similarly, define $Y_{21j} = {}_k(A/I)_A$. Now let $Y_{12} = \bigoplus Y_{12j}$ and $Y_{21} = \bigoplus Y_{21j}$. These are weak direct sums taken over all $j$ in $J$.

We identify $A = Y_{11}$ and $k = Y_{22}$ with the appropriate subalgebras of $Y$ (so that $1_Y = 1_A + 1_k$). We will multiply in $Y$ as with 2 by 2 matrices, using pairings $Y_{mn} \times Y_{np} \to Y_{mp}$ to be defined now.

Products in $Y_{11} = A$ and $Y_{22} = k$ are as usual.

For products $Y_{11} \times Y_{12} \to Y_{12}$, we use the obvious action of $A$ on the left $A$-module $Y_{12} = \bigoplus Y_{12j}$. It is crucial that this idealizes the subset $I$ of $A$, thanks to $I \cdot (A/I) = (0)$. Products $Y_{21} \times Y_{11} \to Y_{21}$ are defined similarly.

For products $Y_{12} \times Y_{22} \to Y_{12}$, we use the obvious action of $k$ on the right $k$-module $Y_{12} = \bigoplus Y_{12j}$. Products $Y_{12} \times Y_{22} \to Y_{12}$ are defined similarly.

The tricky case is $Y_{12} \times Y_{21} \to Y_{11}$. Define the product

$$\Big(\sum_h (a_h + I), \sum_i (c_i + I)\Big) \to \sum_j a_j x_0 c_j$$

as $h$, $i$, $j$ run through $J$. Note that the summation here is defined, because $Y_{12}$ and $Y_{21}$ are weak direct sums. Note also that each term $a_j x_0 c_j$ is well defined, because $x_0$ is in $\operatorname{ann} I$. Moreover, this pairing takes values in $\operatorname{ann} I$, because $\operatorname{ann} I$ is an ideal of $A$ if $I$ is.

We define all products $Y_{21} \times Y_{12} \to Y_{22}$ to be zero.

One readily checks that the $k$-algebra $Y = Y(A, I, x_0, J)$ is associative and unital, contains an isomorphic copy of $A$, namely $Y_{11}$, as a subalgebra, and that $Y_{11}$ contains a copy of $I$ as an ideal.

Thus, if we let $\alpha: I \to A$ and $\eta: I \to Y$ be the obvious inclusions, then we have $(A, \alpha) \prec (Y, \eta)$, thanks to the morphism $(\mathrm{id}, \phi)$: $(A, \alpha) \to (Y, \eta)$. Here $\phi: A \to Y$ is the inclusion with $\phi(A) = Y_{11}$.

It remains to show that $(Y, \eta) \prec (A, \alpha)$ is impossible if $x_0$ and $J$ are chosen appropriately.

Note first that we should not choose $x_0 = 0$. For in this case $Y_{12} \dotplus Y_{21} \dotplus Y_{22}$ is multiplicatively closed and hence an ideal in $Y$. This ideal serves as the kernel of an obvious homomorphism $Y \to Y_{11} = A$ which is nonsingular on $I$.

Thus we choose $x_0 \neq 0$ in $\operatorname{ann} I$ and also choose $J$ to be very large: take $\operatorname{card} J > \operatorname{card} A$. This means that no mapping $\psi$: $Y \to A$, in particular no homomorphism in $k$-Alg, can be injective. (See Comment 1.7.3 below). In fact, $\operatorname{card} J$ is so large that the kernel of a homomorphism $\psi$ must contain a nonzero element of $Y_{12}$ of the form $e_h - e_i$, where $e_h = \sum_j (\delta_{hj} + I)$ in $Y_{12}$. (Here $\delta_{hj}$ is the Kronecker delta.) It follows that the product $(e_h - e_i) \cdot e_h'$ is also in the kernel of $\psi$. Here $e_h'$ is in $Y_{21}$. But $(e_h - e_i) \cdot e_h' = e_h \cdot e_h' = x_0$ in ann $I$. This means that $\psi$ is necessarily singular on the embedded ideal $\eta I$, and hence there is no morphism $(g, \psi): (Y, \eta) \to (A, \alpha)$. (Compare Definition 1.2.1). Thus $(A, \alpha)$ is not extreme. This completes the proof of Theorem 1.7.1.

1.7.3 COMMENT. We suspect that $J$ may be chosen to consist of a single element, that is, that it suffices to use

$$Y(A, I, x_0) = \begin{pmatrix} A & A/I \\ A/I & k \end{pmatrix}.$$

It is easy to prove that there is no *unital* homomorphism $\psi$: $Y(A, I, x_0) \to A$ without any sort of cardinality argument. However, with this $Y$ we were unable to rule out the possibility.

$$A \supset \psi(Y) \supset \psi(A) \supset \psi^2(Y) \supset \cdots \supset I,$$

with $\psi$, $\psi^2$,... injective but not surjective. Thus in the proof we forced $\psi$ to be singular by choosing the index set $J$ very large.

We also mention that it does not appear to be much easier to prove the theorem in the very special case $I^2 = (0)$, $I \simeq k$ as $k$-module.

## 2. THE ALGEBRA OF MULTIPLIERS

### 2.1 *Basic Properties*

Suppose the $k$-algebra $I$ is an ideal in the $k$-algebra $A$. Then each element $a$ in $A$ gives rise to a pair $L_a$, $R_a$ of $k$-linear maps $I \to I$ via $L_a x = ax$, $R_a x = xa$. Moreover, for $a$, $b$ in $A$, associativity forces the relations $L_a R_b x = R_b L_a x$ and $x(L_a y) = (R_a x) y$ for all $x$, $y$ in $I$. Thus, to study the class of ideal embeddings of $I$ into unital $k$-algebras $A$, we are led to consider a family of *pairs* of $k$-linear maps $I \to I$ satisfying relations reminiscent of those just quoted. Let us construct this now.

First, let $k$-lin $I$ be the usual $k$-algebra of all $k$-linear maps $I \to I$ of the $k$-module $I$. Next form the $k$-algebra $(k\text{-lin } I) \oplus (k\text{-lin } I)^{\text{op}}$ consisting of all pairs $\sigma = \langle \sigma', \sigma'' \rangle$ (our standard notation). Here we have the product $\sigma \cdot \tau = \langle \sigma', \sigma'' \rangle \cdot \langle \tau', \tau'' \rangle = \langle \sigma' \tau', \tau'' \sigma'' \rangle$.

2.1.1 DEFINITION. A pair $\sigma = \langle \sigma', \sigma'' \rangle$ in $(k\text{-lin } I) \oplus (k\text{-lin } I)^{\text{op}}$ is a *multiplier* of $I$ if it satisfies

$$\sigma'(xy) = \sigma'(x)y, \quad \sigma''(xy) = x\sigma''(y), \quad x\sigma'(y) = \sigma''(x)y$$

for all $x$, $y$ in $I$. We denote by *$k$-pairs $I$* the set of all multipliers of the $k$-algebra $I$.

One now verifies that $k$-pairs $I$ is a unital associative subalgebra of $(k\text{-lin}\, I) \oplus (k\text{-lin}\, I)^{\mathrm{op}}$. Here $1 = \langle id_I, \mathrm{id}_I \rangle$.

Note that $k$-pairs $I$ is *not* the "algebra of multiplications" generated in $k$-lin $I$ by all left and right multiplications from $I$ [7, p. 46].

We mention at the outset that a binary condition $(\sigma'\tau'')x = (\tau''\sigma')x$, reminiscent of $L_a R_b x = R_b L_a x$, that is $a(xb) = (ax)b$, is *not* incorporated into the definition of $k$-pairs $I$. We will see, not without regrets, that this condition ("cross-commutativity") does not always hold throughout $k$-pairs $I$, nor is there a unique or canonical subalgebra in which it does hold. The point of certain useful theorems will be that this condition does hold within a certain subalgebra of $k$-pairs $I$ for properly conditioned $I$. See 2.5.3 for example.

Let us continue by noting the $k$-algebra homomorphism $\mu: I \to k\text{-pairs}\, I$: $x \to \langle L_x, R_x \rangle$, concocted from the usual left and right regular representations of $I$ on $I$. It is immediate that the kernel of $\mu$ is the two-sided annihilator ann $I$. Moreover, for $\sigma$ in $k$-pairs $I$ and $x$ in $I$, we have $\sigma \cdot \mu(x) = \langle \sigma', \sigma'' \rangle \cdot \langle L_x, R_x \rangle = \langle L_{\sigma'(x)}, R_{\sigma'(x)} \rangle$, while $\mu(x) \cdot \sigma = \langle L_{\sigma''(x)}, R_{\sigma''(x)} \rangle$. Thus the image $\mu I$ is an ideal in $k$-pairs $I$.

Now suppose $I$ happens to be unital with $1 = 1_I$. Then, if $\sigma = \langle \sigma', \sigma'' \rangle$ is a multiplier, we have $\sigma'(x) = \sigma'(1)x = L_{\sigma'(1)}x$, and likewise $\sigma''(x) = R_{\sigma''(1)}x$. But also $\sigma'(1) = 1\sigma'(1) = \sigma''(1)1 = \sigma''(1)$, whence $\sigma = \mu(\sigma'(1))$. In this case, therefore, $\mu: I \to k\text{-pairs}\, I$ is an isomorphism.

We may summarize these observations as follows.

2.1.2 Theorem. *Let $I$ be a $k$-algebra. Then* ($k$-pairs $I$, $\mu$) *is an ideal submersion of $I$ and moreover, is an ideal embedding of $I$ precisely when* ann $I = (0)$. *In particular, if $I$ is unital, then* ($k$-pairs $I$, $\mu$) $\cong$ ($I$, id) *in $\mathscr{IE}(I)$.*

In Theorem 2.1.7 we will observe that, if ann $I = (0)$, then $\mu I$ is in fact an *essential* ideal in $k$-pairs $I$.

Thus we know $k$-pairs $I$ when $I$ is unital. Here are more examples.

2.1.3 Example. If $I^2 = (0)$, then $k\text{-pairs}\, I = (k\text{-lin}\, I) \oplus (k\text{-lin}\, I)^{\mathrm{op}}$, while $\mu$ is the zero map.

2.1.4 Example: General Polynomial Algebras. Let $k$ be as usual, and let $\{x_i\}$ be any nonempty family of algebraically independent indeterminates, possibly noncommuting, over $k$. Let $I$ be the algebra of all polynomials in the $x_i$ with coefficients from $k$ and constant term zero. As usual we have $\mu$: $I \to k\text{-pairs}\, I$, and this monomorphism extends naturally to a monomorphism $\bar{\mu}$: $k[x] \to k\text{-pairs}\, I$, where $k[x] = k[\ldots, x_i, \ldots]$ is the usual algebra of all polynomials in the $x_i$. Note $k[x] \cong k \dotplus I$.

One readily checks that $\bar{\mu}$ is surjective as well. For if $\sigma = \langle \sigma', \sigma'' \rangle$ is in $k$-pairs $I$, then $\sigma'$ and $\sigma''$ are each determined by the images $\sigma'(x_i)$ and $\sigma''(x_i)$.

Now use $x_i\sigma'(x_j) = \sigma''(x_i)x_j$ to prove that $\sigma' = L_u$, $\sigma'' = R_u$, where $u$ is a polynomial in $k[x]$. We conclude $\bar{\mu}$: $k[x] \simeq k$-pairs $I$.

2.1.5 Example: Certain Principal Ideals. We revisit Example 1.6.6. We have $I$, a two-sided ideal in the unital $k$-algebra $A$, with $I = Az = zA$ for a nonzero-divisor $z$ in $I$. One has $\bar{\mu}$: $A \to k$-pairs $I$: $a \to \langle L_a, R_a\rangle$ as usual. (This map will be generalized in 2.2.1.) Observe that $\bar{\mu}$ is a monomorphism. We now prove that $\bar{\mu}$ is onto. First, given $\sigma = \langle\sigma', \sigma''\rangle$ in $k$-pairs $I$, we have $\sigma'(z) = p(\sigma')z$ and $\sigma''(z) = zq(\sigma'')$ for certain elements $p(\sigma')$, $q(\sigma'')$ in $A$. Second, we have $p(\sigma') = q(\sigma'')$, because $zp(\sigma')z = z\sigma'(z) = \sigma''(z)z = zq(\sigma'')z$. Thus we may define $t(\sigma) = p(\sigma') = q(\sigma'')$ in $A$. Third, we have $\sigma' = L_{t(\sigma)}$ and $\sigma'' = R_{t(\sigma)}$ because, for any $c$ in $A$, $z\sigma'(cz) = \sigma''(z)\,cz = zq(\sigma'')\,cz = zt(\sigma)\,cz$, whence $\sigma'(cz) = L_{t(\sigma)}\,cz$, and likewise for $\sigma''$. This proves $\sigma = \bar{\mu}(t(\sigma))$, whence $\bar{\mu}$: $A \simeq k$-pairs $I$ as $k$-algebras. Note that this implies that such an algebra $A$ is uniquely determined by $I$.

2.1.6 Definition. The multiplier $\tau = \langle\tau', \tau''\rangle$ in $k$-pairs $I$ is an *annihilator pair* if $\tau'(I) \subset \operatorname{ann} I$, $\tau''(I) \subset \operatorname{ann} I$, and also $\tau'(I^2) = \tau''(I^2) = (0)$.

The point here is that if arbitrary $\tau'$, $\tau''$ in $k$-lin $I$ satisfy $\tau'(I) \subset \operatorname{ann} I$, $\tau''(I) \subset \operatorname{ann} I$, $\tau'(I^2) = \tau''(I^2) = (0)$, then $\tau = \langle\tau', \tau''\rangle$ is a multiplier. Thus we have a potential source of multipliers (which appear to have little to do with multiplication!).

Observe that if $I > I^2$ and also $\operatorname{ann} I > (0)$, then $k$-pairs $I$ invites nonzero annihilator pairs. In fact, the family of all such is isomorphic with the $k$-module $k\text{-lin}(I/I^2, \operatorname{ann} I) \oplus k\text{-lin}(I/I^2, \operatorname{ann} I)$. Compare Example 2.1.3.

One readily checks that if $\tau$ is an annihilator pair and $x$ is in $I$, then $\tau \cdot \mu(x) = \langle L_{\tau'(x)}, R_{\tau'(x)}\rangle = \langle 0, 0\rangle = 0$; likewise $\mu(x) \cdot \tau = 0$.

These considerations lead to the following description.

2.1.7 Theorem. *The annihilator* $\operatorname{ann}(\mu I;\, k\text{-pairs } I)$ *is the set of all annihilator pairs of I. Consequently,* $\operatorname{ann}(\mu I;\, k\text{-pairs } I) = (0)$ *if and only if* $k\text{-lin}(I/I^2, \operatorname{ann} I) = (0)$ *and, moreover, in these cases $\mu I$ is an essential ideal of $k$-pairs $I$. In particular, if ($k$-pairs $I$, $\mu$) is an ideal embedding, then it is an essential ideal embedding of $I$.*

The next example contrasts with 2.1.4 and 2.1.5.

2.1.8 Example: Strict Upper-Triangular Matrices. Let $k$ be a field and $I =$ the nilpotent algebra of all $n$ by $n$ strict upper triangular matrices over $k$, with $n \geq 3$. Let $A$ be the unital algebra of all $n$ by $n$ upper triangular matrices over $k$. As before, one has a homomorphism $\bar{\mu}$: $A \to k$-pairs $I$ given by $\bar{\mu}(a) = \langle L_a, R_a\rangle$. However, one checks now that $\bar{\mu}A \neq k$-pairs $I$ and that

$\bar{\mu}I \neq$ the radical rad[$k$-pairs $I$]. In fact, this radical contains all annihilator pairs as well as certain other pairs not induced by multiplications from $I$.

We will see in the sequel to this paper that many questions on the radical embedding of $I$ reduce to questions about the radical of $k$-pairs $I$.

2.1.9 EXAMPLE: TRUNCATED POLYNOMIAL IDEALS. Let $k$ be a field and $I$ an $n$-dimensional $k$-algebra with basis of powers $x, x^2, \ldots, x^n$ such that $x^{n+1} = 0$. One checks that $k$-pairs $I = \bar{\mu}(k[x]) \dotplus P$, where $\bar{\mu}$: $k[x] \to k$-pairs $I$: $a \to \langle L_a, R_a \rangle$ is as usual, and $P$ is the one-dimensional ideal consisting of all annihilator pairs $\langle L_{cx^{n-1}}, -R_{cx^{n-1}} \rangle$. Thus $k$-pairs $I$ is commutative and has dimension $n+1$, but is not isomorphic to $k[x] \cong k \dotplus I$ for any $n \geq 1$.

2.1.10 BACKGROUND. At the start of this investigation, when we were seeking an invariant of $I$ which embodied both regular representations, we also considered $\Omega(I) =$ the two-sided idealizer of $\mu(I)$ in ($k$-lin $I$) $\oplus$ ($k$-lin $I)^{\text{op}}$ as a candidate. Clearly $\Omega(I) \supset k$-pairs $I$, and often the two were equal. Also $\Omega(I)$ seemed to have the advantage of being more "functorial" than $k$-pairs $I$, which is defined by mapping conditions. We went on to observe that, in the case of certain important nilpotent $I$, $\Omega(I) > k$-pairs $I$ and, in fact, the mapping conditions mentioned above were surely needed to perform certain desirable constructions which now appear in 2.4 and 2.5. In fact, it became apparent that even more (cross-commutativity) is needed for these constructions. Thus the demise of $\Omega(I)$.

2.1.11 EXERCISE. If $\sigma'$ in $k$-lin $I$ satisfies $\sigma'(xy) = \sigma'(x)y$ for all $x, y$ in $I$, does there exist $\sigma''$ in $k$-lin $I$ such that $\langle \sigma', \sigma'' \rangle$ is a multiplier of $I$? Hint: let $I$ be an $n$-dimensional "column" in the algebra of all $n$ by $n$ matrices.

## 2.2 *A Universal Property of* $k$-pairs $I$

The idea here is that if the homomorphism $\alpha$: $I \to A$ does not destroy too much of $I$, in particular if $(A, \alpha)$ is an ideal embedding, then we may complete the rectangle

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & A \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \alpha^{\#}} \\ I & \xrightarrow[\mu]{} & k\text{-pairs } I \end{array} .$$

2.2.1 THEOREM. *Let* $(A, \alpha)$ *be an ideal submersion of the k-algebra I.*

(i) *If* ker $\alpha$ *is an ideal direct summand of I, then there is a k-algebra homomorphism* $\alpha^{\#}$: $A \to k$-pairs $I$ *with* ker $\alpha^{\#}$ *contained in* ann$(\alpha I; A)$;

(ii) *If also the direct summand* $\ker \alpha$ *is contained in* $\operatorname{ann} I$, *then there is a morphism* $(\mathrm{id}, \alpha^{\#})\colon (A, \alpha) \to (k\text{-pairs}\, I, \mu)$ *of ideal submersions*;

(iii) *Finally, if* $\ker \alpha = (0)$, *that is, if* $(A, \alpha)$ *is an ideal embedding, then the homomorphism* $\alpha^{\#}$ *of* (ii) *is unital.*

Note that none of this requires $\operatorname{ann} I = (0)$.

This result has an important consequence for ideal embeddings of algebras $I$ with $\operatorname{ann} I = (0)$. See 2.3.1 below.

*Proof of Theorem* 2.2.1. We are given that $I = (\ker \alpha) \oplus I_1$, so that every $x = x_0 + x_1$ uniquely in $I$. Moreover, $\alpha(x) = \alpha(x_1)$. Finally, there is an "inverse" map $\gamma\colon \alpha I \to I_1 : \alpha(x) \mapsto x_1$.

To prove (i), suppose $a$ in $A$ and define $\alpha^{\#}(a) = \langle \alpha^{\#}(a)', \alpha^{\#}(a)'' \rangle$, where $\alpha^{\#}(a)'x = (\gamma\lambda_a\alpha)x = \gamma(a\alpha(x))$ and, similarly, $\alpha^{\#}(a)''x = (\gamma\rho_a\alpha)x = \gamma(\alpha(x)a)$. Here $\lambda$ and $\rho$ denote the usual left and right actions of $A$ on $\alpha I$. One readily checks that $\alpha^{\#}$ is a homorphism. It is also necessary to check that $\alpha^{\#}(a)$ is a multiplier of $I$. The major verification goes as follows. We have, for $x$, $y$ in $I$, $x[\alpha^{\#}(a)'\, y] = x_1[(\gamma\lambda_a\, \alpha)\, y_1] = [(\gamma\alpha)x_1] \times [(\gamma\lambda_a\, \alpha)\, y_1] = \gamma([\rho_a\, \alpha x_1]\, \alpha y_1) = [(\gamma\rho_a\, \alpha)x_1]\, y_1 = [\alpha^{\#}(a)''x]\, y$, as required. Note that we used $(\ker \alpha)I_1 = I_1(\ker \alpha) = (0)$ to replace $x$ by $x_1$, etc.

To prove statement (ii) of the theorem, it suffices to show $(\alpha^{\#}\, \alpha)z = \mu(z)$ for all $z$ in $I$. But $(\alpha^{\#}\, \alpha)z = \langle \gamma\lambda_{\alpha z}\alpha, \gamma\rho_{\alpha z}\alpha \rangle$ and, for $x$ in $I$, $(\gamma\lambda_{\alpha z}\alpha)x = (\gamma\alpha)\, zx = zx = L_z x$. Likewise, $(\gamma\rho_{\alpha z}\alpha)x = R_z x$. This yields statement (ii).

The proof of (iii) is immediate.

## 2.3 *Extreme Embeddings when* $\operatorname{ann} I = (0)$

This result was previewed in 1.7.2, but requires $k$-pairs $I$ for its statement.

2.3.1 Theorem. *Let* $\operatorname{ann} I = (0)$. *Then* $(k\text{-pairs}\, I, \mu)$ *is the unique (up to equivalence) extreme element of* $\mathscr{I}\mathscr{E}(I)$.

*Proof.* Since $\operatorname{ann} I = (0)$, we have from 2.1.2 that $(k\text{-pairs}\, I, \mu)$ is in $\mathscr{I}\mathscr{E}(I)$. If also $(A, \alpha)$ is in $\mathscr{I}\mathscr{E}(I)$, then 2.2.1 (ii) shows $(A, \alpha) \prec (k\text{-pairs}\, I, \mu)$. Done.

Contrast Theorem 1.7.1, in which a nonzero $\operatorname{ann} I$ guaranteed there were no extreme elements in $\mathscr{I}\mathscr{E}(I)$. Thus the global structure of the class $\mathscr{I}\mathscr{E}(I)$ with respect to the transitive relation $\prec$ (or of the partially ordered quotient class $\mathscr{I}\mathscr{E}(I)/\sim$) depends dramatically on the annihilator of $I$, as follows:

(i) if $\operatorname{ann} I = (0)$, then every ideal embedding $(A, \alpha)$ is sandwiched between "adjunction of a unity" and the algebra of multipliers,

$$(k \dotplus I, \iota) \prec (A, \alpha) \prec (k\text{-pairs}\, I, \mu).$$

In other words, the partially ordered class $\mathscr{I}\mathscr{E}(I)/\sim$ has a unique "least" and a unique "greatest" element with respect to $\prec$;

(ii) if $\operatorname{ann} I \neq (0)$, then every element of $\mathscr{I}\mathscr{E}(I)/\sim$ is "strictly less" than some other element. Of course the equivalence class of $(k \dotplus I, \iota)$ again serves as a least element in $\mathscr{I}\mathscr{E}(I)/\sim$.

This answers Questions 1.6.2 for $\mathscr{I}\mathscr{E}(I)$. Now we turn to $\mathscr{S}\mathscr{D}(I)$.

## 2.4 *Natural Sums and Cross-Commutativity*

Thus far we have used the multiplier algebra $k$-pairs $I$ to settle a problem about extreme elements in the class $\mathscr{I}\mathscr{E}(I)$ of all ideal embeddings of $I$ (Theorem 2.3.1). We will have several further uses for $k$-pairs $I$, both now and in the sequel to this paper, as we seek extreme semidirect or radical embeddings of $I$. These will frequently involve "natural sums."

2.4.1 Definition. Let $A$ be any $k$-subalgebra, not necessarily unital, of $k$-pairs $I$. The *natural sum* of $A$ and $I$ is the external semidirect sum $A \dotplus I$ with coordinatewise addition and multiplication given by

$$(\sigma, x)(\tau, y) = (\sigma \cdot \tau, \sigma'(y) + \tau''(x) + xy)$$

for $\sigma$, $\tau$ in $A$ and $x$, $y$ in $I$.

*Caution.* The natural sum may fail to be unital. More important, it *must* fail to be associative unless $\sigma'\tau'' = \tau''\sigma'$ for all $\sigma$, $\tau$ in $A$. That is, the algebra $I$ may fail to be a $(A, A)$-bimodule in the usual sense. See the discussion following Definition 2.1.1. One contrasts the possible failure of $\sigma'\tau'' = \tau''\sigma'$ as maps of $I$ with the fact that $\sigma \cdot (\mu(x) \cdot \tau) = (\sigma \cdot \mu(x)) \cdot \tau$ *within* $k$-pairs $I$ itself. A brief reflection will convince the reader that $\operatorname{ann} I$ is involved here (and also the condition $I^2 \neq I$; see 2.4.3 below).

2.4.2 Definition. The subalgebra $A$ of $k$-pairs $I$ is *cross-commutative* if $\sigma'\tau'' = \tau''\sigma'$ for all $\sigma$, $\tau$ in $A$.

It is immediate that $A$ is cross-commutative if and only if the corresponding natural sum $A \dotplus I$ is associative.

Now we measure the failure of cross-commutativity by examining the commutator $[\sigma', \tau''] = \sigma'\tau'' - \tau''\sigma'$ as a $k$-linear mapping $I \to I$.

2.4.3 Lemma. *Let $\sigma$, $\tau$ be in* $k$-pairs $I$. *Then*

(i) $[\sigma', \tau''](I) \subset \operatorname{ann} I$, and

(ii) $[\sigma', \tau''](I^2) = (0)$.

*Thus $\langle [\sigma', \tau''], [\tau', \sigma''] \rangle$ is an annihilator pair in the multiplier algebra* $k$-pairs $I$.

*Proof.* (i) To prove that $[\sigma', \tau'']x$ is in the *left* annihilator of $I$, observe that, for all $y$ in $I$, $([\sigma', \tau'']x)y = \sigma'(\tau''(x))y - \tau''(\sigma'(x))y = \sigma'(\tau''(x)y) - \sigma'(x)\tau'(y) = \sigma'(x\tau'(y)) - \sigma'(x)\,\tau'(y) = \sigma'(x)\,\tau'(y) - \sigma'(x)\,\tau'(y) = 0$. Likewise for the right annihilator. This proves (i).

(ii) For $xy$ in $I^2$, one has $[\sigma', \tau'']\,xy = \sigma'(\tau''(xy)) - \tau''(\sigma'(xy)) = \sigma'(x\tau'(y)) - \tau''(\sigma'(x)y) = \sigma'(x)\,\tau'(y) - \sigma'(x)\,\tau'(y) = 0$, as asserted.

Here is an immediate consequence.

2.4.4 THEOREM. *If* $k$-lin $(I/I^2, \operatorname{ann} I) = (0)$, *then* $k$-pairs $I$ *and hence each of its subalgebras* $A$ *is cross-commutative. Consequently all natural sums* $A \dotplus I$ *are associative algebras (though not necessarily unital).*

In 2.5 we will apply this theorem and other observations to get extreme semidirect embeddings of appropriate $I$. In the sequel to this paper we will obtain sufficient conditions for associativity of natural sums $A \dotplus I$ for certain $A$ and certain nilpotent $I$ (to which 2.4.4 surely does not apply).

For the moment, however, we are content to give the above partial answer to the question, "Which algebras $k$-pairs $I$ are cross-commutative?" We see in particular that if $I$ is unital, or a generalized polynomial algebra (Example 2.1.4) or a principal ideal as in Example 2.1.5, then $k$-pairs $I$ is cross-commutative. In fact, if the obvious map $k \dotplus I \to k$-pairs $I$ is an epimorphism, then $k$-pairs $I$ is easily seen to be cross-commutative.

Here is an example of cross-commutativity with $I \neq I^2$, $\operatorname{ann} I \neq (0)$, and $k \dotplus I$ not isomorphic to $k$-pairs $I$.

2.4.5 EXAMPLE. Let $I$ be the truncated polynomial ideal of Example 2.1.9. Observe that if $\sigma = \langle \sigma', \sigma'' \rangle$ is in $k$-pairs $I$, then $\sigma'$ and $\sigma''$ are induced by a left or right multiplication within the associative algebra $k \dotplus I$. The point here is that even if $\sigma'(I) \subset \operatorname{ann} I$ and $\sigma'(I^2) = (0)$, we still have $\sigma'$ given by a left multiplication. It follows that $k$-pairs $I$ is cross-commutative.

Here is an important case where $k$-pairs $I$ is *not* cross-commutative.

2.4.6 EXAMPLE. Let $I$ be the strict upper triangular $n$ by $n$ matrices of Example 2.1.8. Let $\{x_{ij}\}$ with $1 \leqslant i < j \leqslant n$ be the usual basis. Take $n \geqslant 4$. Consider $\sigma'$ in $k$-lin $I$ determined by $\sigma'(x_{23}) = x_{1n}$ and $\sigma'(x_{ij}) = 0$ if $i \neq 2$, $j \neq 3$. Then $\sigma = \langle \sigma', 0 \rangle$ is in $k$-pairs $I$, as is $\langle L_e, R_e \rangle$ where $e = x_{33}$ is a non-strict upper-triangular matrix. Since $\sigma'(R_e x_{23}) = x_{1n}$ while $R_e(\sigma'(x_{23})) = 0$, we have $k$-pairs $I$ failing to be cross-commutative.

## 2.5 *The Multiplier Algebra and Semidirect Embeddings*

We continue to seek extreme semidirect ideal embeddings. If the natural sum $A \dotplus I$ is associative (that is, if $A$ is cross-commutative) and unital, then we immediately obtain a "natural" semidirect ideal embedding $(A \dot{+} I, j_A)$ in $\mathscr{SD}(I)$ by defining $j(x) = j_A(x) = (0, x)$ for $x$ in $I$.

Let us observe that this construction is not so special as it might seem, and that $k$-pairs $I$ carries decisive information about semidirect embeddings. For if $(A, \alpha)$ is in $\mathscr{SD}(I)$, say $A = S \dot{+} \alpha I$, then we have $\alpha^{\#}: A \to k\text{-pairs}\, I$ (see Theorem 2.2.1). One checks that $\alpha^{\#}(S)$ is a cross-commutative subalgebra of $k$-pairs $I$ (in fact, $\alpha^{\#}(S) \cong S/\mathrm{ann}(\alpha I; S)$) and, moreover, that the associative algebra $\alpha^{\#}(S) \dot{+} I$, a natural sum, is unital (in fact, its unity element is $1 = (\alpha^{\#}(u_S), u_I)$ where $1_A = u_S + u_I$ uniquely in $A$). Thus we have produced a semidirect ideal embedding $(\alpha^{\#}(S) \dot{+} I, j)$ with $j = j_{\alpha^{\#}(S)}$.

Moreover, note that $(A, \alpha) < (\alpha^{\#}(S) \dot{+} I, j)$ in $\mathscr{SD}(I)$, thanks to the morphism $(\mathrm{id}, \hat{\alpha})$, where $\hat{\alpha}: A = S \dot{+} \alpha I \to \alpha^{\#}(S) \dot{+} I$ is defined by $\hat{\alpha}(s + \alpha(x)) = (\alpha^{\#}(s), x)$.

2.5.1 LEMMA. *If $(A, \alpha)$ is in $\mathscr{SD}(I)$ with $A = S \dot{+} \alpha I$, then $(A, \alpha) <$ $(\alpha^{\#}(S) \dot{+} I, j)$ in $\mathscr{SD}(I)$.*

The moral here is that in seeking extreme elements of $\mathscr{SD}(I)$, we need only consider natural sums.

Note that we made good use here of the decomposition $A = S \dot{+} \alpha I$. Nothing quite so sharp could be proved for an arbitrary ideal embedding in $\mathscr{IE}(I)$.

Now we present two applications of Lemma 2.5.1 to the question of extreme semidirect ideal embeddings. Namely, (1) they always exist, thanks to Zorn's lemma (contrast Theorem 1.7.1, which showed that extreme *ideal* embeddings do not exist if $\mathrm{ann}\, I \neq (0)$), and (2) under certain conditions the only extreme semidirect embedding is into the natural sum $(k\text{-pairs}\, I) \dot{+} I$.

2.5.2 THEOREM. *Let $I$ be an arbitrary associative $k$-algebra. Then every semidirect ideal embedding $(A, \alpha)$ of $I$ is $<$ than a natural semidirect sum ideal embedding of the form $(A \dot{+} I, j)$ which is extreme in $\mathscr{SD}(I)$.*

Thus both Questions 1.6.2 have affirmative answers in the class $\mathscr{C} = \mathscr{SD}(I)$ for every $I$.

Note that we make no assertions as to the uniqueness of extreme semidirect embeddings in the general case.

*Proof.* (sketch). Apply Zorn's lemma to the nonempty family of cross-commutative subalgebras of $k$-pairs $I$, partially ordered by inclusion (and inductive!). Deduce the existence of maximal crosscommutative subalgebras $A$, which in fact contain the unity of $k$-pairs $I$ by maximality. Thus we obtain semidirect ideal embeddings into natural sums $A \dot{+} I$, as in the statement. That each of these embeddings is in fact an extreme element in $\mathscr{SD}(I)$ is a consequence of Lemma 2.5.1 and the maximality of $A$. Checking this completes the proof.

Now we point out certain algebras $I$ (there will be others) with a unique (up to equivalence) extreme element in $\mathscr{S}\mathscr{D}(I)$.

2.5.3 THEOREM. *If* k-pairs $I$ *is cross-commutative, then the natural semidirect embedding* $((k\text{-pairs}\, I) \dagger I, j)$ *is the unique (up to equivalence) extreme element of* $\mathscr{S}\mathscr{D}(I)$. *In particular, this is the case if either* $\operatorname{ann} I = (0)$ *or* $I^2 = I$.

See 2.4 for examples of algebras $I$ to which this theorem applies.

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